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SECOND ORDER PERTURBED RANDOM DIFFERENTIAL EQUATION

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ABSTRACT : In this paper ,we investigate the second order nonlinear perturbed functional random differential equation. Prove the existence of random solution through Leray- Schauder fixed point theorem.

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1. STATEMENT OF PROBLEM

Let R denote the real line. Let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in R for some real numbers r and a with $r > 0$ and $a > 0$. Let $C = C(I_0, R)$ denote the space of all condtinuous real valued functions on I_0 equipped with the $\|\cdot\|_C$ defined by

$$\| x \|_C = \sup_{t \in I_0} | x(t) |$$

Given a measurable space (Ω, A) and a given a history function $\phi : \Omega \to C(I_0, R)$,

We consider the following second order perturbed functional random differential equation (PFRDE)

$$\begin{aligned} x''(t,\omega) &= f(t, x_t(\omega), \omega) + g(t, x_t(\omega), \omega) + h(t, x_t(\omega), \omega),\ a.e. t \in I, \omega \in \Omega. \\ x(t,\omega) &= \phi_0(t,\omega), x'(t,\omega) = \phi_1(t,\omega)\ , t \in I_0. \end{aligned} \tag{1.1}$$

For all $\omega \in \Omega$, where $f, g : I \times C \times \Omega \to R$ and the function $x_t : \Omega \to C(I_0, R)$ is defined by $x_t(\omega) = x(t+\theta, \omega), -r \le \theta \le 0$, for each $t \in I$.

By a random solution of PFRDE (1.1) we mean a measurable function $x:\Omega \to C(J,R)\cap C(I_0,R)\cap AC(I,R)$ that satisfies the equation (1.1) on J, wher $AC(J,R)$ is the space of all absolutely continuous real valued functions on J.

The study of nonlinear perturbed differential equation and nonlinear integral equations of mixed type has bcen made by Burton and Kirk [1] and Dhage [3] by using the fixed point theorems of Leray-Schauder type.In this paper we shall use a random version of the Leray-Schauder type principle Dhage [3] and study the nonlinear initial value problems of perturded functional random differential equations of second order for different aspects of the solutions under suitable conditions.

2. EXISTENCE RESULTS

let (Ω, A) denote a measurable space. X a separable Banach space. Let β_X be a σ-algebra of all Borel subsets of X.

Let $T:X \to X$. T is called a contraction if there exists a constant $a<1$ such that, $\|Tx-Ty\| \le \alpha \|x-y\|$ for all $x,y \in X$. A random operator $T:\Omega \times X \to X$ is called contraction (resp. compact totally bounded and completely continuous) if $T(\omega)$ is contraction (resp. compact, totally bounded and completely continuous) for each $\omega \in \Omega$.

We use the following the fixed point theorems[3].

Theorem 2.1. Let $A,B:\Omega \times X \to X$ be two random operator satisfying for each $\omega \in \Omega$,

(a) $A(\omega)$ is contraction,

(b) $B(\omega)$ is completely continuous and

(c) The set $\varepsilon = \{u:\Omega \to X \mid A(\omega)u + B(\omega)u = \alpha u\}$ is bounded for all $\alpha > 1$. Then the random equation.

$$A(\omega)x + B(\omega)x = x \tag{2.1}$$

has a random solution.

Theorem 2.2. Let $A,B:X \to X$ be two operators such that :

(a) A is linear and bounded, and there exists a $\mathrm{P} \in N$ such that A^{P} is a nonlinear

(b) B is completely continuous.

Then either

(i) The operator equation $Ax + \lambda Bx = x$ has a solution for $\lambda = 1$ or

(ii) The set $\varepsilon = \{u \in X \mid Au + \lambda Bu - u, 0 < \lambda < 1\}$ is unbounded.

Theorem 2.3. Let $A,B:\Omega \times X \to X$ be two random operator satisfying for each $\omega \in \Omega$,

(a) $A(\omega)$ is linear and bounded, and there exists a $\mathrm{P} \in N$ such that A^{P} is a nonlinear contraction,

(b) $B(\omega)$ is completely continuous and

(c) The set $\varepsilon = \{u \in X \mid A(\omega)u + \lambda(\omega)u = u\}$ is bounded for every measurable function $\lambda : \Omega \to R$ with $0 < \lambda(\omega) < 1$.

Then the operator equation

$$A(\omega)x + B(\omega)x = x$$

has a random solution.

As a theorem 2.2, we obtain.

Corollary 2.1. Let $A, B : \Omega \times X \to X$ be two random operator satisfying for each $\omega \in \Omega$,

(a) $A(\omega)$ is contraction,

(b) $B(\omega)$ is completely continuous and

(c) The set $\varepsilon = \{u \in X \mid A(\omega)u + \lambda B(\omega)u = u\}$ is bounded for each $\lambda \in (0,1)$.

Then the random equation (2.1) has a random solution.

We need the following definition.

Definition 2.1. A function $\beta : J \times C \times \Omega \to R$ is said to be ω-caratheodory if for each $\omega \in \Omega$.

(i) $t \to f(t, x, \omega)$ is measurable for all $x \in C$. and

(ii) $x \to f(t, x, \omega)$ is continuous for almost everywhere $t \in J$.

Further a ω-caratheodory function β is called L^1 - Caratheodory if

(iii) For each real number $k > 0$ there exists a function $h_k : \Omega \to L^1(J, R)$ such that

$$|\beta(t, x, \omega)| \leq h_k(t, \omega).a.e.t \in J$$

For all $x \in C$ with $\| x(\omega) \|_C \leq k$.

3.MAIN RESULT

We consider the following set of hypotheses.

(A_1) The function $\omega \to f(t, x, \omega)$ is measurable for all $t \in I$ and $x \in C$.

(A_2) The function $t \to f(t, x, \omega)$ is continuous for each $\omega \in \Omega$, and there exists a

Function $\alpha : \Omega \to L^1(J, R)$, with $\| \alpha(\omega) \|_{L^1} < 1$, such that for each $\omega \in \Omega$

$$| f(t, x, \omega) - f(t, y, \omega) | \leq \alpha(t, \omega) - y(\omega) \|_C \ a.e.t \in I$$

for all $x, y \in C$.

(A_3) The function $\omega \to g(t,x,\omega)$ is measurable for all $t \in I$ and $x \in C$.

(A_4) The function ***g*** is L^1_ω - random caratheodory.

(A_5) There exists a function $\gamma : \Omega \to L^1(J,R)$ with $\gamma(t,\omega) > 0$ a.e. $t \in J$ and a

Continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that.

$|g(t,x,\omega) + h(t,x_t(\omega),\omega)| \le \gamma(t,\omega)\psi(\| x(\omega) \|_C)$ a. e. $t \in I$ for all $x \in C$.

Theorem 3.1. Assume that hypotheses (A_1) – (A_5) hold. Further suppose that

$\| \alpha(\omega) \|_{L^1} < 1$ and

$$\int_a^\infty \frac{dz}{z + \psi(z)} > \| \gamma(\omega) \|_{L^1} \tag{3.1}$$

Where

$$c_0(\omega) = \| \phi(\omega) \|_C + \int_0^t | f(s,0,\omega) | ds \text{ and } \gamma(s,\omega) = \max\{\alpha(s,\omega).\gamma(s,\omega)\}.$$

Then the PFRDE (1.1) has a solution on *J*.

Proof. Let $X = C(J,R)$. Now the FRDE (1.1) is equivalent to the random integral equation(RIE)

$$x(t,\omega) = \phi_0(t,\omega) + \phi_1(t,\omega)t + \int_0^t (t-s) f(t,x_s(\omega)ds + \int_0^t (t-s)g(t,x_s(\omega),\omega)ds \int_0^t (t-s)h(t,x_s(\omega),\omega)ds, \; a.e. t \in I,$$

Define two operators $A, B : J \times C \times \Omega \to X$ by

$$A(\omega)x(t) = \int_0^t (t-s) f(t,x_s(\omega)ds, \quad a.e. t \in I, \tag{3.2}$$

and

$$B(\omega)x(t) = \phi_0(t,\omega) + \phi_1(t,\omega)t + \int_0^t (t-s)g(t,x_s(\omega),\omega)ds + \int_0^t (t-s)g(t,x_s(\omega),\omega)ds, \; a.e. t \in I,$$

(3.3)

Then the problems of finding the random solution of the perturbed FRDE (1.1) is just reduced to finding the random solution of random equation $A(\omega)x(t) + B(\omega)x(t) = x(t), t \in I$ in *X*. This further implies that the random fixed points of the operator equation $A(\omega)x + B(\omega)x = x$ are the random solution of the FRDE (1.1) on *J*. We shall show that the operators $A(\omega)$ and $B(\omega)$ satisfying all the conditions of Theorem 2.1

Step I : First we show that $A(\omega)$ and $B(\omega)$ are random operators on *X*. Since

$$\omega \to f(t,x_t(\omega),\omega)$$

is measurable for each $t \in I$ and $x \in C$, and the integral on the right hand side of the equation (3.2) is the limit of the finite sum of measurable function, the function

$$\omega \to \int_0^t f(t, x_s(\omega), \omega) ds$$

is measurable. Hence the operator $A(\omega)$ is a random operator on X.

Again the function $\omega \to \phi(t, \omega)$ is measurable for each $t \in I_0$ and the integral

$$\omega \to \int_0^t g(t, x_s(\omega), \omega) ds, \omega \to \int_0^t h(t, x_s(\omega), \omega) ds$$

are measurable, therefore and the sum

$$\phi_0(t, \omega) + \phi_1(t, \omega) t + \int_0^t (t-s) g(t, x_s(\omega), \omega) ds + \int_0^t (t-s) h(t, x_s(\omega), \omega) ds, \ a.e. t \in I,$$

is measurable in $\omega \in \Omega$ for each $t \in I$. Hence the operator $B(\omega)$ is a random operator on X.

Step II : Next we show that $A(\omega)$ is a contraction random operator on X. Let $x, y \in X$. Then by (A_2),

$$\begin{aligned} | A(\omega)x(t) - A(\omega)y(t) | &= | \int_0^t f(s, x_t(\omega), \omega) ds - \int_0^t (s, y_t(\omega), \omega) ds | \\ &\le \alpha(t, \omega) \| x_t(\omega) - y_t(\omega) \|_C \\ &\le \| \alpha(\omega) \|_{L^1} \| x(\omega) - y(\omega) \|_C \end{aligned}$$

Taking Supremum over t, we obtain

$$\| A(\omega)x(t) - A(\omega)y(t) \| \le \| \alpha(\omega) \|_{L^1} \| x(\omega) - y(\omega) \|_C$$

For all $x, y \in X$ and $\omega \in \Omega$, where $\| \alpha(\omega) \|_{L^1} < 1$. This shows that $A(\omega)$ is a contraction random operator on X.

Step III : Now we shall show that the random operator $B(\omega)$ is completely continuous on X. First we show that $B(\omega)$ is continuous on X. Using the dominated convergence theorem and the continuity of the function $g(t, x, \omega)$ in x, it follows that

$$\begin{aligned} B(\omega)x_n(t) &= \phi_0(t, \omega) + \phi_1(t, \omega) t + \int_0^t (t-s) g(t, x_n(\omega), \omega) ds + \int_0^t (t-s) h(t, x_n(\omega), \omega) ds \\ &\to \phi_0(t, \omega) + \phi_1(t, \omega) t + \int_0^t (t-s) g(t, x(\omega), \omega) ds + \int_0^t (t-s) h(t, x(\omega), \omega) ds \\ &= B(\omega)x(t). \end{aligned}$$

For all $t \in I$.

Similarly,

$$| B(\omega)x_n(t) | = \phi(t, \omega) = B(\omega)x(t)$$

For all $t \in I_0$. This shows that $B(\omega)$ is continuous random operator on *X*.

Next we show that $B(\omega)$ is a totally bounded random operator on *X*. To finish, it is enough to prove that $\{B(\omega)x_n : n \in N\}$ is uniformly bounded and equicontinuous set in *X*. Suppose that $x_n(t,\omega)$ is a bounded sequence in *X*. Then there is a real number $r > 0$ such that $x_n(t,\omega) \le r$, $\forall n \in N$. Now

$$|B(\omega)x_n(t)| \le \max\{|\phi(0,\omega)| + |\phi_1(t,\omega)t|\} + \int_0^t (t-s)|g(s,x_n(s+\theta,\omega),\omega)|ds + \int_0^t (t-s)|g(s,x_n(s+\theta,\omega),\omega)|ds$$
$$\le \|\phi(\omega)\|_C + \int_0^t h_r(s,\omega)ds$$
$$\le \|\phi(\omega)\|_C + \int_0^a h_r(s,\omega)ds$$
$$\le \|\phi(\omega)\|_C + \|h_r(\omega)\|_{L^1}.$$

Taking supremum over *t*, we obtain

$$\|B(\omega)x_n\| \le \|\phi(\omega)\|_C + \|h_r(\omega)\|_{L^1}$$

Which shows that $\{B(\omega)x_n : x \in N\}$ is uniformly bounded set in *X*.

Next we show that the set $\{B(\omega)x_n : x \in N\}$ is an equicontinuous set. Let $t,\tau \in I$. Then

$$|B(\omega)x(t) - B(\omega)x(\tau)| \le \left|\int_0^t g(s,x_s(\omega),\omega)ds - \int_0^\tau g(s,x_s(\omega),\omega)ds\right| + \left|\int_0^t h(s,x_s(\omega),\omega)ds - \int_0^\tau h(s,x_s(\omega),\omega)ds\right|$$
$$\le \left|\int_\tau^t |g(s,x_s(\omega),\omega)|ds\right| + \left|\int_\tau^t |h(s,x_s(\omega),\omega)|ds\right|$$
$$\le \left|\int_\tau^t h_r(s,\omega)ds\right|$$
$$\le |\mathrm{P}(t,\omega) - \mathrm{P}(\tau,\omega)|.$$

Where $\rho(t,\omega) = \int_0^t h_r(s,\omega)ds.$

Since P is continuous on *I*, it is uniformly continuous on *I*. Therefore

$$|B(\omega)x(t) - B(\omega)x(\tau)| \to 0 \text{ as } t \to \tau.$$

Again let $t,\tau \in I_0$. Then we have

$$|B(\omega)x(t) - B(\omega)x(\tau)| = |\phi(t,\omega) - \phi(\tau,\omega)| \to 0 \text{ as } t \to \tau.$$

Similarly if $t \in I$ and $\tau \in I_0$ then we obtain

$$|B(\omega)x(t) - B(\omega)x(\tau)| = \left|(\phi_0(t,\omega) + \phi_1(t,\omega)t) - (\phi_0(\tau,\omega) + \phi_1(\tau,\omega)\tau) + \int_0^t (t-s)g(s,x_s(\omega),\omega)ds) + \int_0^t (t-s)h(s,x_s(\omega),\omega)ds)\right|$$
$$\le |\phi_1(\tau,\omega)\tau - \phi_1(t,\omega)t| + \int_0^t |g(s,x_s(\omega),\omega)|ds + \int_0^t |h(s,x_s(\omega),\omega)|ds$$
$$\le |\phi(\tau,\omega) - \phi(t,\omega)| + \int_0^t h_r(s,\omega)ds.$$

Now if $|t-\tau| \to 0,$ thus we have $\tau \to 0.$ as $\tau \to 0.$ so by continuity of ϕ and the integral, it follows that.

$$\phi_1(\tau,\omega)\tau - \phi_1(t,\omega)t| \text{ as } \tau \to 0$$

And

$$\int^{t} h_r(s,\omega)ds \to 0 \text{ as } t \to 0.$$

Therefore in all three cases we have

$$|B(\omega)x(t) - B(\omega)x(\tau)| \to 0 \text{ as } t \to \tau.$$

Hence the set $\{B(\omega)x_n : x \in N\}$ is an equicontinuous in *X*. Thus the random operator $B(\omega)$ is completely continuous in view of Arezela-Ascoli Theorem.

Finally we show that the hypothesis (c) of Theorem 2.1 hold.

Let $u \in \varepsilon$ be arbitray. Then we have $A(\omega)u(t) + B(\omega)u(t) = \lambda u(t,\omega); \lambda > 1$ for all $t \in J$. Therefore

$$u(t,\omega) = \lambda^{-1}[A(\omega)u(t) + B(\omega)u(t)]$$

For $t \in J$. Hence

$$|u(t,\omega)| = \lambda^{-1}\left(\phi_0(t,\omega) + \phi_1(t,\omega)t + \int_0^t (t-s)f(t,x_s(\omega)ds + \int_0^t (t-s)g(t,x_s(\omega),\omega)ds + \int_0^t (t-s)h(t,x_s(\omega),\omega)ds\right)$$

Hence if $t \in I$.

$$|u(t,\omega)| \leq |\lambda^{-1}| \max\{|\phi(0,\omega)|.|\phi(t,\omega)\}$$
$$+|\lambda^{-1}|\left|\int_0^t (t-s)f(s,u_s(\omega),\omega)ds\right| + |\lambda^{-1}|\left|\int_0^t (t-s)g(s,n_s(\omega),\omega)ds\right| + |\lambda^{-1}|\left|\int_0^t (t-s)h(s,n_s(\omega),\omega)ds\right|$$
$$\leq \|\phi(\omega)\|_C + \int_0^t |(t-s)||f(s,u_s(\omega),\omega)|ds + \int_0^t |(t-s)||g(s,u_s(\omega),\omega)|ds + \int_0^t |(t-s)||h(s,u_s(\omega),\omega)|ds$$

$$\leq \|\phi(\omega)\|_C + \int_0^t |(t-s)||f(s,n_s(\omega),\omega) - f(s,0,\omega)|ds$$
$$+\int_0^t |(t-s)||f(s,0,\omega)|ds + \int_0^t |(t-s)|\gamma(t,\omega)\psi(\|u_s(\omega)\|_C)ds$$
$$\leq \|\phi(\omega)\|_C + \int_0^t \alpha(s,\omega)\|u_s(\omega)\|_{C^{ds}}$$
$$+\int_0^t |f(s,0,\omega)|ds + \int_0^t \gamma(t,\omega)\psi(\|u_s(\omega)\|_C)ds$$
$$\leq c_0(\omega) + \int_0^t \gamma(s,\omega)[\|u_s(\omega)\|_C + \psi(\|u_s(\omega)\|_C)]ds.$$

Set $\omega(t,\omega) = \max_s \in [-r,t]|u(s,\omega)|$. Then $|u(t,\omega)| \leq \omega(t,\omega), \forall t \in J$ and $\omega \in \Omega$, and there is a $t^* \in [-r,t]$ such that

$$u(t,\omega) = |u(t^*,\omega)| = \max_{s\in|-rd} |u(t,\omega)|$$

For all $\omega \in \Omega$. Therefore for any $t \in I$ we get

$$\omega(t,\omega) = c_0(\omega) + \int_0^{t^*} \gamma(s,\omega)[\| u_s(\omega) \|_C + \psi(\| u_s(\omega) \|_C]ds$$
$$\leq c_0(\omega) + \int_0^t \gamma(s,\omega)[\omega(s,\omega) + \psi(\omega(s,\omega))]ds.$$

Let

$$m(t,s) = c_0(\omega) + \int_0^t \gamma(s,\omega)[\omega(s,\omega) + \psi(\omega(s,\omega))ds. \ t \in I.$$

Then we have $\omega(t,\omega) \leq m(t,\omega), \forall t \in I$ and $\omega \in \Omega$ and $m(0,x) = c_0(\omega)$.

Differentiating w.r.t., *t* yields

$$m'(t,\omega) = \gamma(t,\omega)[\omega(t,\omega) + \psi(\omega(t,\omega))]$$
$$\leq \gamma(t,\omega)[m(t,\omega) + \psi(m(t,\omega))], t \in I.$$

Hence from above inequality we obtain

$$\frac{m'(t,\omega)}{m(t,\omega) + \psi(m(t,\omega))} \leq \gamma(t,\omega), t \in I.$$

Integrating from 0 to *t* gives

$$\int_0^t \frac{m'(s,\omega)}{m(s,\omega) + \psi(m(s,\omega)} ds \leq \int_0^t \gamma(s,\omega)ds$$

By change of the variable, we obtain

$$\int_{c_0(\omega)}^{m(s,\omega)} \frac{dz}{z + \psi(z)} \leq \int_0^t \gamma(s,\omega)ds \leq \int_0^a \gamma(s,\omega)ds < \int_{c_0(\omega)}^{\infty} \frac{dz}{z + \psi(z)}$$

This implies that there exists a constant $M(\omega) > 0$ such that

$$m(t,\omega) \leq M(\omega), \forall t \in J \text{ and } \omega \in \Omega.$$

Then we have

$$| u(t,\omega) | \leq \omega(t,\omega) \leq m(t,\omega) \leq M(\omega), \forall t \in J \text{ and } \omega \in \Omega.$$

Then the set ε is bounded. Hence an application of Theorem 2.1 yields that the PFRDE (1.1) has a solution on *J*. This completes the proof.

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